has been necessary because of the key relation that exists between $H_1(s)$ and $H_2(s)$ as stated by Lemma 1 [1] when $H_1 = FP$ and $H_2 = P$ and by Lemma 2 in the present case. In the general case, all we can say at the present time is that if H_1 has no unstable hidden modes and the relation given by Lemma 2 is satisfied by H_1 and H_2, the existence of N, V, and W satisfying the above equations is sufficient for RPIS to be solvable.

The particular formulation used in Section II may be useful in parameterizing all stabilizing controllers through the rational function K used in [1] to solve RPIS. This could possibly lead to a better understanding of minimal-order stabilizing controllers.

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On Alternative Methodologies for the Design of Robust Linear Multivariable Regulators

H. G. KWATNY AND K. C. KALNITSKY

Abstract—**This paper presents two synthesis algorithms which embody the two major variants of the numerous methodologies which have been proposed for the design of multivariable linear regulators which exhibit the property of disturbance rejection with or without additional robustness qualities. It is shown that these two procedures generally lead to substantively different compensator structures.**

I. Introduction

During the past decade a number of techniques have been proposed for the design of multivariable linear regulators enjoying the properties of disturbance rejection and, in some cases, structural stability [1]–[12], [14], [15], [18], [26]–[30]. Many of these procedures have been fairly widely applied [3], [13], [16], [17], [19], [20], [30], and in view of the interest which has been evidenced by theoretician and practitioner alike, it is clear that with the development of these concepts the day of application of "modern" multivariable control theory is at hand. It appears that although many investigators have independently evolved their own specific design methodologies, these can be grouped into two distinct variations—those employing estimates of (possibly artificial) disturbance states, and those employing dynamic error augmentation. Examples of the former have been proposed by Kwatny *et al.* [15], [18], Balchen *et al.* [30], Parker [29], Sebakhy and Wonham [26], and Francis [27]. Examples of the latter have been proposed by Davison *et al.* [6]–[11], Young and Williams [12], and Calovic and Cuk [14].

In the present authors' work both types of design procedures have been applied—specifically those methods of Kwatny *et al.* and Davison *et al.* Experience has shown that closed-loop transient behavior can be significantly different even when the designs are carried out with the intent of attaining the same performance requirements. The question naturally arises as to whether these differences come about because of the inherent latitude the designer has at various points within the design processes or whether they are, in fact, due to fundamental differences in structure. This paper reports on studies intended to provide at least a partial answer to this question.

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In Section II, the regulator problem and the robust regulator problem are defined as they will be discussed in this paper. The adopted formulation is somewhat less general than can be treated and that can be found discussed in several of the papers cited above. Nevertheless, this choice has been made in order to avoid obscuring the main ideas with a host of nonessential technical detail. Section III presents two basic design algorithms which typify the essential variants to be found in the references. Section IV identifies the fundamental difference between these methodologies and correlates this result with the classical compensator design techniques. Section V presents a brief concluding statement.

II. The Regulation Problem and Compensator Structure

This paper is concerned with a linear time-invariant system defined by the equations

$$
\begin{aligned}
\dot{x} &= Ax + E\omega + Bu \\
\dot{\omega} &= Z\omega \\
y &= Cx + F\omega \\
\bar{y} &= G\omega \\
e &= y - \bar{y}
\end{aligned}
\tag{1}
$$

where x is an n-dimensional plant state vector, y is an r-dimensional output vector, $\bar{y}$ is an r-dimensional reference output, u is an m-dimensional input vector, ω is a q-dimensional vector representing a combined state for the exogenous disturbance and output reference, and e is an r-dimensional error vector. In what follows it is assumed that (A,B) is controllable and (C,A) is observable and that B and C are of full rank. With some restriction in generality it is assumed that the composite pair $\left\{[\,C \mid (F-G)\,], \left[\begin{array}{c|c} A & E \\ \hline U & Z \end{array}\right]\right\}$ is observable. Where this condition is required it can usually be relaxed to detectability. The discussion in Francis [27] on this assumption is pertinent. The *regulator problem* is the construction of a feedback controller such that the closed-loop system—excluding the disturbance states ω, is stable (internal stability) and $e(t) \rightarrow 0$ as $t \rightarrow \infty$ for all initial states (output regulation).

A *robust* (or *structually stable*) solution of the regulator problem has the desirable property that closed-loop stability and output regulation are preserved under specified classes of perturbations of plant and controller parameters.

Numerous researchers have studied the regulator problem from various viewpoints in recent years [1]–[12], [14], [15], [18], [26]–[30]. The notion of robust solutions of the regulator problem appears to have originated with Davison [9] and has been further examined by Davison and Goldenberg [11], Pearson *et al.* [12], Francis and Wonham [25], Sebakhy and Wonham [26], and Francis [27].

Necessary and sufficient conditions for the existence of a solution to the regulator problem and the robust regulator problem have been stated by several authors, notably Davison [9], Davison and Goldenberg [11], Francis and Wonham [25], and Francis [27]. These conditions are summarized for the problem as stated above in the following theorem.

Theorem 1: A necessary and sufficient condition for the existance of a solution to the regulator problem is that the following conditions hold:

1) (A,B) is stabilizable;

2) (C,A) is detectable;

3) There exists an $n \times q$ matrix X and an $m \times q$ matrix U satisfying the relations

$$
\begin{aligned}
AX - XZ + BU &= E \\
CX &= F - G.
\end{aligned}
$$

A necessary and sufficient condition for the solution to the robust regulator problem is obtained if 3) is replaced by

4) $\operatorname{rank}\begin{bmatrix} A-\lambda_i & B \\ C & 0 \end{bmatrix} = n + r$, for each λ_i in the spectrum of Z.

Proof of Theorem 1 for the regulator problem is given by Francis [27].

The matrices required in the third condition were utilized constructively in solution of the regulator problem by Kwatny *et al.* [15], [18]. For the robust case a proof is given by Francis, Sebakhy, and Wonham [37], and in Francis [27], where arbitrary perturbations are allowed in the plant parameters A, E, B. Davison and Goldenberg [11] provide a proof where arbitrary perturbations are allowed in the plant parameters A, B, C. Weaker conditions may be required if a smaller set of parameter variations are allowed [35]. If the output y is not the error vector e, then an additional condition is that e is readable from y. The fourth requirement of Theorem 1 implies that $m \geq r$ and that none of the invariant zeros of the system correspond to disturbance eigenvalues.

The statement of necessary and sufficient conditions for the robust regulator is facilitated by introducing the notion of an internal model (Francis and Wonham [25]). A matrix A incorporates an internal model of degree l of a matrix A_2 if the minimal polynomial of A_2 divides precisely l invariant factors of A. An internal model of degree l is an l-fold reduplication in A of the maximum cyclic component of A_2. In the context of the regulator problem the following terminology is occasionally employed. A incorporates a "weak" internal model of A_2 if $1 \leq l < r$ and a "strong" internal model of A_2 if $l \geq r$. If A incorporates an internal model of A_2 and is the system matrix of a system S, then it is meaningful to speak of observability and controllability of the internal model in the sense that all modes corresponding to the internal model are, respectively, observable or controllable. The internal model is an essential aspect of the regulator problem as is obvious from the following theorem. The term synthesis is used to denote a plant-compensator combination.

Theorem 2: (Necessity of the internal model [25], [36]). A synthesis exhibits internal stability and output regulation only if the compensator-plant combination incorporates a weak internal model of Z which is controllable by e and observable by y. A synthesis is robust with respect to arbitrary perturbations of E only if the compensator incorporates a strong internal model which is controllable by e and observable by u.

Again it is noted that for a more general output set y than is being considered here, it is also necessary that e be readable from y. If this is the case, Theorem 2 remains valid provided that the additional information carried by y pertains to the plant and is unavailable from e [25].

Theorem 3: (Sufficiency of the internal model [25]). Suppose a synthesis exhibits internal stability and the compensator incorporates a weak internal model of Z which is controllable by e and observable by u. Then the synthesis exhibits output regulation. If the compensator incorporates a strong internal model of Z which is controllable by e and observable by u, then the synthesis is robust with respect to plant parameters A, E, B and all compensator parameters except that part of the compensator dynamics which contains the internal model of Z.

For a discussion of what happens when there are perturbations of that part of the compensator dynamics which contains the internal model, see Francis and Wonham [25] and Davison and Goldenberg [11].

III. Synthesis Procedures

Of the numerous synthesis procedures proposed in the literature [1]–[12], [14], [15], [18], [26]–[30], all appear to fall into two distinct categories. Loosely speaking, one class is based on the feedback of estimates of the disturbance states, and the other on the feedback of the states of a dynamic system driven by the error vector. Included in the former category are the methodologies of Kwatny *et al.* [15], [18], Balchen *et al.* [30], Parker [29], Sebakhy and Wonham [26], and Francis [27]. The latter category includes the procedures of Davison *et al.* [6]–[11], Young and Williams [12], and Calovic and Cuk [14]. As will be seen below, the compensators are, in general, substantively different. However, under appropriate circumstances, depending upon plant parameters and/or designer choices, both classes of synthesis procedures will result in identical compensators. In fact, if the methodology[1] proposed by Johnson in his pioneering works [1]–[5] is specialized to the case of linear time-invariant systems, then the class of plants which can be dealt with is restricted in such a way that both classes of synthesis procedures will produce identical compensators.

[1]Johnson describes three alternative points of view regarding disturbance accommodations. It is the "counteraction" mode which is referred to here as it is the only one in the spirit of the other procedures considered herein.

The two synthesis procedures described below typify the two classes of procedures referred to.

Synthesis Procedure A

Consider the composite system

$$\dot{x}^* = A^*x^* + B^*u \qquad e = C^*x^* \tag{2}$$

where

$$A^* = \begin{bmatrix} A & E \\ 0 & Z \end{bmatrix}, \quad B^* = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad C^* = [\, C \;\vdots\; (F-G) \,], \quad x^* = \begin{bmatrix} x \\ \omega \end{bmatrix}.$$

The algorithm proceeds as follows:

Step 1: Find a state feedback control law

$$u = -Mx^* = -M_1x - M_2\omega \tag{3}$$

such that

a) $A - BM_1$ is stable (internal stability), and

b) $e(t) \to 0$ as $t \to \infty$ for all initial states $x^*(0)$ (output regulation). This is equivalent to the requirement that

$$C^* e^{(A^* - B^*M)t} \equiv 0.$$

Step 2: Design a minimal order observer for the composite system which will take the form

$$\dot{\xi} = \Delta\xi + \Sigma e + \Omega u \qquad \hat{x}^* = W_1\xi + W_2 e \tag{4}$$

where the parameter matrices Δ, Σ, Ω, W_1, W_2 are obtained in the usual way. Note the $\dim(\xi) = n + q - r$.

Step 3: Implement the control

$$u = -M\hat{x}^* = -M_1\hat{x} - M_2\hat{\omega}. \tag{5}$$

Explicit procedures for carrying out Step 1 have been given by Kwatny *et al.* [15], [18], and Sebahky and Wonham [26]. This problem has been extensively studied in general by Bhattacharyya, Pearson and Wonham [21], Bhattacharyya and Pearson [22], and Wonham and Pearson [24]. Step 2, of course, is routine [32].

Synthesis Procedure B

Consider the system defined by (1).

Step 1: Define the q-dimensional, error driven dynamical system

$$\dot{\eta} = Z\eta + Je \tag{6}$$

where J is chosen so that (J, Z) is controllable.

Step 2: Consider the composite system

$$\begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A & 0 \\ JC & Z \end{bmatrix} \begin{bmatrix} x \\ \eta \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \cdot u. \tag{7}$$

Find a state feedback control law

$$u = -K_1x - K_2\eta \tag{8}$$

such that the closed-loop system is stable.

Step 3: Design a minimal order observer for the system (1) which will take the form

$$\dot{\sigma} = \overline{\Lambda}\sigma + \overline{\Sigma}y + \overline{\Omega}u \qquad \hat{x} = \overline{W}_1\sigma + \overline{W}_2 y. \tag{9}$$

Note that $\dim(\sigma) = n - r$.

Step 4: Implement the control

$$u=-K_1\hat{x}-K_2\eta. \tag{10}$$

It should be noted that satisfaction of the conditions of Theorem 1 along with the assumptions following (1) allow both procedures to be carried out. Moreover, in both cases the synthesis exhibits internal stability by construction and each procedure incorporates an internal model of Z in the compensator. In procedure B the internal model is incorporated through the construction of Step 1. Also by construction, the internal model is controllable by e. The fact that the composite system (7) is stabilized in Step 2 and Z is unstable guarantees that the internal model is observable by u. With procedure A, the incorporation of an internal model is not obvious, although it is to be anticipated by virtue of the construction. A proof of incorporation of an internal model which is controllable by e and observable by u is given by Sebakhy and Wonham [26], and Francis [27].

Whether or not the compensator is a robust solution to the regulator problem depends on the structure of Z. Generally, Z is constructed so that it contains a weak or strong, as desired, internal model of a primary matrix disturbance Z_0. If only a weak internal model is required then it is sufficient to take $Z=Z_0$. If a strong internal model is required then a procedure of outward extension may be used to define Z as described by Sebakhy and Wonham [26], and Francis [27]. Note that Z_0 may already contain a strong internal model of itself. The compensator can therefore be endowed with the required type of internal model of the primary disturbance dynamics by appropriate definition of Z.

Certain elementary properties of the resultant compensators should be noted. Both procedures result in compensators with dimension $n+q-r$. Thus, the resultant closed-loop synthesis will have dimension $2n+q-r$. The closed-loop poles are located as follows:

Synthesis Procedure A: n eigenvalues of $A-BM_1$

$n+q-r$ eigenvalues of Δ.

Synthesis Procedure B: $n+q$ eigenvalues of

$$\begin{bmatrix} A-BK_1 & -BK_2 \\ JC & 0 \end{bmatrix}$$

$n-r$ eigenvalues of $\bar{\Delta}$.

Since the eigenvalue locations of each synthesis come from two distinct subproblems and these subproblems carry different dimensions in the two procedures, there will be circumstances under which no selection of design parameters can result in a specified common set of closed-loop eigenvalues. These exceptional situations arise only because, when n,r are both odd, synthesis procedure A cannot meet a pole specification which does not include at least two real poles, but synthesis procedure B can. Similarly, when n is even and r is odd synthesis procedure B cannot meet a pole specification which does not include at least two real poles, but synthesis procedure A can. This appears to be a rather trivial distinction since closed-loop pole assignment is generally not such a precise design objective that an acceptable specification cannot be found which is compatible to both procedures.

The compensator structures resulting from the two synthesis procedures are summarized in Fig. 1.

A final point worth mentioning is that there are variants of both synthesis A [23] and synthesis B [11] which are not based on the use of observers for the construction of estimates of plant or disturbance states.

Differences in Compensator Characteristics

The block diagrams of Fig. 1 exhibit the structural forms of multivariable servomechanisms designed via the disturbance estimation (synthesis procedure A) and error augmentation (synthesis procedure B) approaches. These illustrations are suggestive of the two principal compensator configurations of classical single-input-single-output (SISO) control theory. The type A design results in a series compensation structure, while the type B design results in a feedback compensation structure of the most common minor within major loop configuration [33], [34]. Indeed, these analogies go beyond mere suggestion. This notion will be developed through the determination of the closed-loop

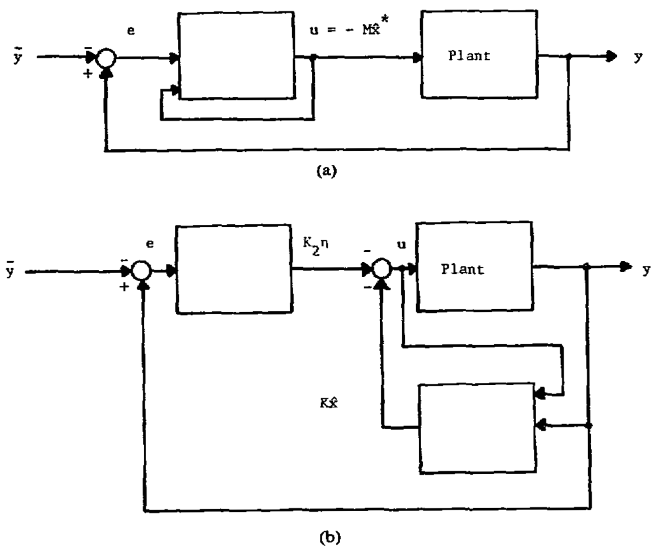

Fig. 1. Compensator structure. (a) Synthesis procedure A. (b) Synthesis procedure B.

system zeros. In particular, it is most convenient to examine the error response to reference signal excitation.

The analysis is facilitated by the following lemma.

Lemma 1: Consider the identity feedback system

$$\begin{aligned} \dot{x}&=Ax+Be,\\ y&=Cx,\\ e&=y-\bar{y}. \end{aligned} \tag{11}$$

The error response invariant zeros are the eigenvalues of A. Moreover, if (B,A) is controllable and (A,C) is observable, then the error response transmission zeros are the eigenvalues of A.

Proof: The system matrix with input $\bar{y}$ and output e is

$$P(s)=\begin{bmatrix} (A+BC)-sI & -B \\ C & -I \end{bmatrix}.$$

It follows from [31, Theorem 1] that the invariant zeros are the eigenvalues of the matrix

$$(\Delta+BC)-BC=A.$$

If the controllability and observability conditions hold, as postulated, then these carry over to the feedback system with input $\bar{y}$ and output e. Thus, all of the invariant zeros are transmission zeros.

Lemma 1 can now be used in the proof of the following.

Theorem 4: The error response invariant zeros for the synthesis of type A consist of: the n plant poles and the $n+q-r$ compensator poles (the eigenvalues of $\Delta-\Omega MW_1$ and q of which correspond to the eigenvalues of Z). The error response transmission zeros consist of: the n plant poles and those of the $n+q-r$ compensator poles which are controllable by e and observable by u. The q compensator poles corresponding to Z are always transmission zeros.

The error response invariant zeros for the synthesis of type B consist of: n modified plant poles corresponding to the eigenvalues of $(A-BK_1)$, the $n-r$ observer eigenvalues corresponding to the eigenvalues of $\bar{\Delta}$ and the q eigenvalues of Z. The transmission zeros consist of: the eigenvalues of $(A-BK_1)$ and the eigenvalues of Z.

Proof: That the invariant zeros are as stated requires application of Lemma 1 in both cases. Determination of the transmission zeros requires only exclusion of those invariant zeros which correspond to uncontrollable or unobservable modes of the closed-loop system. In both cases, the fact that the eigenvalues of Z are included in the set of transmission zeros follows from the observability and controllability of the internal model. The plant is assumed controllable and observable so the plant poles are retained as transmission zeros in case A. In the case of synthesis B it is well known that the plant with the state-feedback with observer configuration retains the controllability and observability of the plant state, but the observer states are uncontrollable [32].

The conclusions of Theorem 4 imply that even if the closed-loop pole assignments are identical, the error responses to command signals will in general be different for a synthesis of type A and a synthesis of type B by virtue of the fact that the closed-loop zero patterns will be different. Moreover, the zero patterns of the two designs are clearly analogous to classical series and feedback compensated servomechanisms, respectively. That is, in series compensated single-input-single-output systems, the error response zeros are the inner loop poles and the cascade compensator poles [34]. It is possible under special conditions to use the design flexibility inherent in both synthesis procedures at various stages to obtain identical zero patterns. In general, however, this is *not* possible.

V. Conclusions

It has been noted that of the numerous multivariable linear regulator design methodologies which have evolved during the past decade, there appear to be two distinct categories. In order to examine the nature of any substantive differences inherent in the compensators which are the product of these design procedures, two synthesis algorithms have been described which, it is claimed, embody the essence of the alternative methods. As a means for effective study of the distinguishing characteristics of the several design algorithms which have been cited, it has been found convenient to discuss these procedures in a scenario which is perhaps less general than the conditions under which any one of them may be applicable. Thus, many of the algorithms are usable in more general situations and specifically allow a broader class of measurement sets and plant models.

The study has shown that although it is usually possible to meet a specific closed-loop pole assignment objective with either of the two basic algorithms, the resultant closed-loop zero patterns will typically be different and thus the response to reference signals will be different. Moreover, it has been shown that the zero patterns that arise in the system error response are identical to those which arise in the two classical SISO compensator configurations, i.e., series and feedback compensation. A discussion of the relative merits of type A versus type B has been avoided at this stage. Indeed, in the SISO case there is considerable lore in this regard [33], [34]. Besides the fact that the zero patterns will be different, there are important implications with regard to hardware implementation and sensitivity. Much of the classical discussion is imprecise and it may be that a deeper appreciation of these alternative configurations can be achieved in the modern multivariable context.

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Desensitizing Constant Gain Feedback Linear Regulators

P. J. FLEMING

***Abstract*—A two-stage process is proposed for the design of low-sensitivity constant gain feedback linear regulators. In the first stage nominal parameter values are assumed and a model response is obtained. Plant parameter variations are taken into account in the second stage, and a sensitivity reduction algorithm is described in which a performance index which includes a model-following term is to be minimized. The computer solution of the feedback matrix is obtained using a gradient search method, and a fourth-order aircraft flight control example illustrates the design's capabilities.**

I. Introduction

A two-stage design process is proposed. In the first stage a model response is obtained which describes the desired dynamical behavior of the regulator. Determination of this response may be carried out by design methods such as optimal linear-quadratic control, pole assign-

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